SOME POLYNOMIAL INVARIANTS OF A FAMILY OF GRAPHS

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ABSTRACT. *We give the general form of the Tutte polynomial of a family of positive-signed connected planar graphs. Then we give general formulas of the Jones polynomial of some very interesting families of alternating knots and links that correspond to these planar graphs. We also give general forms of the flow, reliability, and chromatic polynomials of these graphs. Finally, we give some useful combinatorial information by evaluating the Tutte polynomial at some special points.*

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1. INTRODUCTION

The Tutte polynomial was introduced by W. T. Tutte in 1954 in [22] as a generalization of chromatic polynomials studied by Birkhoff [1] and Whitney [25]. This graph invariant became popular because of its universal property that any multiplicative graph invariant with a deletion/contraction reduction must be an evaluation of it, and because of its applications in computer science, engineering, optimization, physics, biology, and knot theory.

In 1985, V. F. R. Jones revolutionized knot theory by defining the Jones polynomial as a knot invariant via Von Neumann algebras [12]. However, in 1987 L. H. Kauffman introduced in [14] a state-sum model construction of the Jones polynomial that was purely combinatorial and remarkably simple; we follow this construction.

Our primary motivation to study the Tutte polynomial came from the remarkable connection between the Tutte and the Jones polynomials that up to a sign and multiplication by a power of t the Jones polynomial $V_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_G(-t,-t^{-1})$[19, 16, 11].

This paper is organized as follows: In Section 2 we shall give some basic notions about graphs and knots along with definitions of the Tutte and the Jones polynomials. In this section we shall also give the relation between graphs and knots, and the relation between the Tutte and the Jones. Then main results will be given in Section 3.

2. Preliminary Notions

2.1. Basic Concepts of Graphs

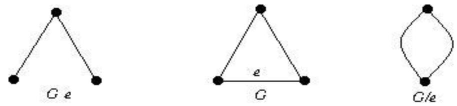
A *graph* G is an ordered pair (V,E) of disjoint sets such that E is a subset of the set V^2 of unordered pairs of V; the set $V=V(G)$ is the set of *vertices* and $E=E(G)$ is the set of *edges*. An edge (x,y) is said to join the vertices x and y, and is denoted by xy; the vertices x and y are the end vertices of this edge. If $xy \in E(G)$, then x and y are said to be *adjacent* vertices of G, and the vertices x and y are *incident* to the edge xy. Two edges are adjacent if they have exactly one common end vertex. A graph $G'=(V',E')$ is said to be a subgraph of $G=(V,E)$ if $V' \subset V$ and $E' \subset E$. If G' contains all edges of G that join two vertices in V' then G' is said to be the *subgraph* induced or spanned by V', and is denoted by $G[V']$. Thus, a subgraph G' of G is an induced subgraph if $G'=G[V(G')]$. If $V=V'$, then G' is said to be a spanning subgraph of G. Two graphs are *isomorphic* if there is a correspondence between their vertex sets that preserves adjacency. Thus, $G=(V,E)$ is isomorphic to $G'=(V',E')$, denoted by $G \cong G'$, if there is a bijection $\phi: V \to V'$ such that $xy \in E$ if and only if $\phi(xy) \in E'$. The dual notion of a cycle is that of a cut or cocycle. If $\{V_1,V_2\}$ is a partition of the vertex set, and the set C, consisting of those edges with one end in V_1 and one end in V_2, is not empty, then C is called a *cut*. A cycle with one edge is called a *loop* and a cocycle with one edge is called a *bridge*. We refer to an edge that is neither a loop nor a bridge as ordinary. A graph is *connected* if there is a path from one vertex to any other vertex of the graph. A connected subgraph of a graph G is called a *component* of G. We denote by $k(G)$ the number of connected components of a graph G, and by $c(G)$ the number of non-trivial connected components, that is the number of connected components not counting isolated vertices. A graph is k-*connected* if at least k vertices must be removed to disconnect the graph. A *tree* is a connected graph without cycles. A *forest* is a graph whose connected components are all trees. Observe that a loop in a connected graph can be characterized as an edge that is in no spanning tree, while a bridge is an edge that is in every spanning tree. A graph is *planar* if it can be drawn in the plane without crossings of edges.

A drawing of a graph in the plane separates the plane into regions called *faces*. Every plane graph G has a dual graph, G^*, formed by assigning a vertex of G^*, to each face of G and joining two vertices of G^* by k edges if and only if the corresponding faces of G share k edges in their boundaries. If G is connected, then $(G^*)=G$.

A *graph invariant* is a function f on the collection of all graphs such that $f(G_2)=f(G_2)$ whenever $G_1 \cong G_2$. A graph polynomial is a graph invariant whose images lie in a polynomial ring.

2.2 The Tutte Polynomial

The following two operations are essential to understand the Tutte polynomial definition for a graph G. These are: *edge deletion* denoted by $G'=G-e$ and *edge contraction* $G''=G/e$.



Definition 2.1. [21, 22, 23]

The Tutte polynomial of a graph G is a two-variable polynomial $T_G(x,y)$ defined as follows:

$$T_G(x,y)=\begin{cases}1 & If\ E\ is\ empty\\ xT\left(\frac{G}{e}\right) & if\ e\ is\ bridge\\ yT(G-e) & if\ e\ is\ loop\\ T\left(\frac{G}{e}\right)-T(G-e) & if\ e\ is\ niether\ bridge\ nor\ loop\end{cases}$$

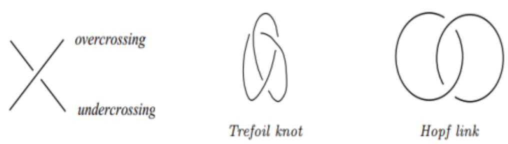
Example.

$$T(\ )=T(\ )+T(\ )=x^3+T()=x^3+T(\ )+T(\ )=x^3+x^2+T(\ )+T(\ )=x^3+x^2+x+y.$$

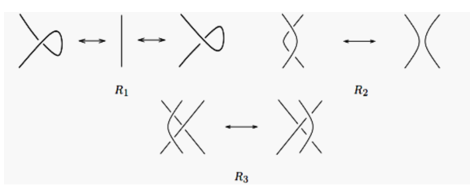
2.3 Basic Concepts of Knots

A *knot* is a circle embedded in $\mathbb{R}^3$, and a link is an embedding of a union of such circles; each circle is called a *component* of the link. A *knot* is a one component link. We shall often use the term link for both knots and links. Links are usually studied via projecting them on a plan; a projection with extra information of overcrossing and undercrossing is called the *link diagram*.

Two links are called *isotopic* if one of them can be transformed to the other by a diffeomorphism of the ambient space $\mathbb{R}^3$ onto itself. A fundamental result about the isotopic link diagrams is: Two unoriented links L_1 and L_2 are



equivalent if and only if a diagram of L_1 can be transformed into a diagram of L_2 by a finite sequence of ambient isotopies of the plane and local (Reidemeister) moves of the following three types [19]:



The set of all links that are equivalent to a link L is called a class of L. By a link L we shall always mean a class of the link L.

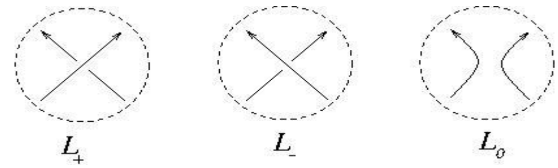
2.4 The Jones Polynomials

The main question of knot theory is which two links are equivalent and which are not? To address this question one needs a knot invariant, a function that gives one value on all links that belong to a single class and gives different values (but not always) on links that belong to different classes. We are concerned with the knot invariant, the Jones polynomial.

Definition 2.3. [12, 13, 14] The *Jones polynomial* $V_k(t)$ of an oriented link L is a Laurent polynomial in the variable $\sqrt{t}$ satisfying the skein relation

$$t^{-1}-tV_{L_-}=\left(t^{1/2}-t^{-1/2}\right)V_{L_0}$$

and that the value of the unknot is 1. Here L_+, L_- and L_0 are three oriented links having diagrams that are isotopic everywhere except at one crossing where they differ as in the figure:



Example. The Jones polynomials of the Hopf link and the trefoil knot are:

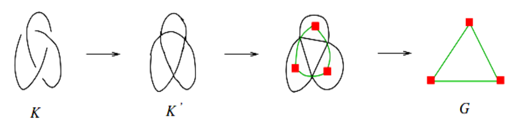
$$V\left(\ \right)=t^{-5/2}-t^{-1/2}$$

$$V\left(\ \right)=-t^4+t^3+t$$

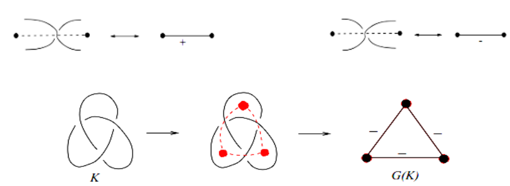
2.5 A Connection between Knots and Graphs

Corresponding to every connected link diagram we can find a connected signed planar graph and vice versa. The process is as follows: Suppose K is a knot

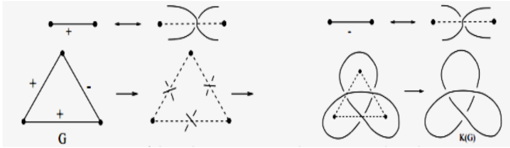
and K' its projection. The projection K' divides the plane into several regions. Starting with the outermost region, we can color the regions either white or black. By our convention, we color the outermost region white. Now, we color the regions so that on either side of an edge the colors never agree.



Next, choose a vertex in each black region. If two black regions R and R' have common crossing points $c_1, c_2 \ldots . c_n$ then we connect the selected vertices of R and R' by simple edges that pass through $c_1, c_2 \ldots . c_n$ and lie in these two black regions. In this way, we obtain from K' a plane graph G [17]. However, in order for the plane graph to embody some of the characteristics of the knot, we need to use the regular diagram rather than the projection. So, we need to consider the under- and over-crossings. To this end, we assign to each edge of G either the sign $+$ or $-$ as you can see in the following figure.



A signed plane graph that has been formed by means of the above process is said to be the graph of the knot K [17]. Conversely, corresponding to a connected signed planar graph, we can find a connected planar link diagram. The construction is clear from the following figure.



The fundamental combinatorial result connecting knots and graphs is:

Theorem 2.4. [16] The collection of connected planar link diagrams is in one-to-one correspondence with the collection of connected signed planar graphs.

2.6. Connection between the Tutte and the Jones polynomials

The primary motivation to study the Tutte polynomial came from the following remarkable connection between the Tutte and the Jones polynomials.

Theorem 2.5. (Thistlethwaite's) [20, 16, 11] Up to a sign and multiplication by a power of t the Jones polynomial $V_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_G(-t, -t-1)$.

For positive-signed connected graphs, we have the precise connection:

Theorem 2.6. [2] Let G be the positive-signed connected planar graph of an alternating oriented link diagram L. Then the Jones polynomial of the link L is $V_L(t) = (-1)^{wr(L)} t^{\frac{(b(L)-a(L)+3wr(L))}{4}} T_G(-t, -t^{-1})$ where $a(L)$ is the number of vertices in G, $b(L)$ is the number of vertices in the dual of G, and *wr(L)* is the writhe of L.

Remark 2.7. In this paper, we shall compute Jones polynomials of links that correspond only to positive-signed graphs.

Example: Corresponding to the positive-signed graph G: we receive the right handed trefoil knot L: It is easy to check that $V\left(\;, t\right) = -t^4 + t^3 + t$ and $V\left(\;, x, y\right) = x^2 + x + y$. Further note that the number of vertices in G is 3, nuber of vertices in dual in G is 2, and writhe of L is 3. Now notice that

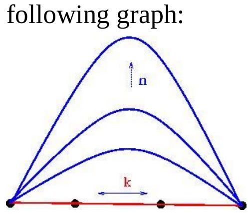
$$V\left(\;, t\right) = (-1)^3 t^{\frac{2-3+3(3)}{4}} T(\;, -t, -t^{-1})$$
$$= -t^2(t^2 - t - t^{-1})$$

3. Main Results

In this section the Tutte, Jones, flow, chromatic, and reliability polynomials are given. Some evaluations of the Tutte polynomial along with the homology of the planar graphs are also presented.

3.1 The Tutte Polynomial

In this section we give the general form of the Tutte polynomial of the following graph:



For reference purposes, we denote this graph by $G_{k.n}$ which is made up of a k vertices bus with n additional edges joined at the end vertices of the bus.

Theorem 3.1. [8] If G and G' are graphs then $T(G \sqcup G') = T(G)T(G')$ and $T(G * G')\, T(G)T(G')$, where $G \sqcup G'$ is the disjoint union of G and G' and $G * G'$ is formed by identifying a vertex of G and a vertex of G' into a single vertex.

Lemma 3.2. The Tutte polynomial of the cycle C_n is $T_{C_n}(x, y) = \sum_{i=1}^{n-1} x^i + y$.

Proof. We prove it by induction on n. The Tutte polynomials of the cycles C_3, C_4, and C_5 are:

$$T\left(\; \right) = x^2 + x + y,$$

$$T(\;) = T(\;) + T(\;) = x^3 + x^2 + x + y.$$ and

$$T\left(\; \right) = T\left(\; \right) + T\left(\; \right) = x^4 + x^3 + x^2 + x + y.$$

Suppose the result holds for $n = k-1$, i.e., $T_{C_{k-1}}(x, y) = \sum_{i=1}^{k-1} x^i + y)$ Now taking a cycle C_k with k vertices, we have $T(C_k) = T(C_k - e) + \mathrm{T}(C_k - 1)$, where $C_k - e$ means an edge is deleted from C_k. Using $T(C_k - e) = x^k$ and the inductive step, we get

$$T(C_k) = x^k + \sum_{i=1}^{k-2}(x^i + y) = \sum_{i=1}^{k-1}(x^i + y).$$

Theorem 3.3: The Tutte Polynomial of $G_{k,n}$ is

$$T_{G_{k,n}}(x, y) = x^{k-1} + \left(\sum_{i=1}^{k-2} x^i + y\right) \sum_{j=0}^{n-1} y^j$$
$$= \boldsymbol{x^{k-1}} + \left(\frac{x^{k-1}-x}{x-1} + y\right)\left(\frac{y^n-1}{y-1}\right)$$

for $n \geq 1, k \geq 1$.

Proof: We proof it by induction on n by keeping k fixed.

For $n = 1$, we have

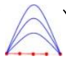
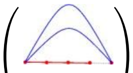
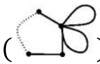
$$T(\;) = T(\;) + T\left(\; \right)$$
$$= x^{k-1} + \sum_{i=1}^{k-2} x^i + y$$

To get a clear picture, we also give Tutte polynomial for $n = 2$ and $n = 3$:

For $n = 2$, we have

$$T\left(\; \right) = T(\;) + T\left(\; \right)$$
$$= x^{k-1} + \sum_{i=1}^{k-2} x^i + y + T(\;)$$
$$= x^{k-1} + \sum_{i=1}^{k-2} x^i + y + y\left(\sum_{i=1}^{k-2} x^i + y\right)$$
$$= x^{k-1} + \left(\sum_{i=1}^{k-2} x^i + y\right)(1 + y)$$

For $n = 3$, we get

$T\Big($  $\Big) = T\Big($  $\Big) + T\Big($  $\Big)$

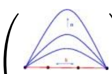
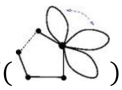
$$= x^{k-1} + \left(\sum_{i=1}^{k-2} x^i + y\right)(1+y) + \; y^2\left(\sum_{i=1}^{k-2} x^i + y\right)$$

$$= x^{k-1} + \left(\sum_{i=1}^{k-2} x^i + y\right)(1+y+y^2)$$

Suppose the result hold for $n = N$ and $k \geq 4$, that is

$$T_{G_{k,N}}(x,y) = x^{k-1} + \left(\sum_{i=1}^{N-2} x^i + y\right)\sum_{j=0}^{N-1} y^j$$

If $n = N+1$, then

$T\Big($  $\Big) = T_{G_{k,N}} + T\Big($  $\Big)$

$$= \left[x^{k-1} + \left(\sum_{i=1}^{k-2} x^i + y\right)\sum_{j=0}^{N-1} y^j\right] + y^N\left(\sum_{i=1}^{k-2} x^i + y\right)$$

$$= x^{k-1} + \left(\sum_{i=1}^{k-2} x^i + y\right)\sum_{j=0}^{N} y^j$$

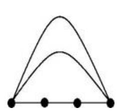
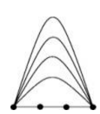
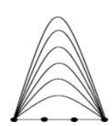
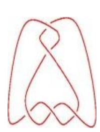
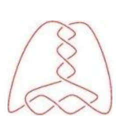
$= \boldsymbol{x^{k-1}} + \left(\frac{x^{k-1}-x}{x-1} + y\right)\left(\frac{y^{N+1}-1}{y-1}\right)$, which is the required result.

3.2 The Jones Polynomial

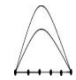
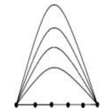
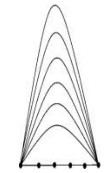
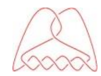
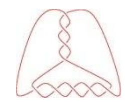
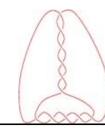
The alternating links L that correspond to the graph $G_{k,n}$ fall into two categories, the 1-componet links (when n is even) and the 2-component links (when n is odd). Depending on k and n, we receive the four cases:

Case I: (k *and* n *are even*.)

When both k and n are even, we receive 1-component links (means simply knots); these knots along with the corresponding graphs are given in the following tables. When $k = 4$

n	2	4	6
G			
L			
$a(L)$	4	4	4
$b(L)$	3	5	7
$wr(L)$	-5	-7	-9

When $k = 6$

n	2	4	6
G			
L			
$a(L)$	6	6	6
$b(L)$	3	5	7
$wr(L)$	-7	-9	-11

Proposition 3.4: The Jones polynomial of the alternating link L that corresponding to the planar graph $G_{k,n}$, when both k and n are even , is $V_L(t) = t^{\frac{-n}{2}} + \frac{t^{\frac{-3}{2}n-k+1}}{(1+t)^2}[(t^k - t^2 - t - 1)(1-t^n)]$.

Proof: We prove it by specializing the Tutte polynomial of the graph $G_{k,n}$ using Theorem 2.3, which says that

$$V_L(t) = (-1)^{wr(L)} t^{\frac{b(L)-a(L)+3wr(L)}{4}} T_{G_{k,n}}(-t,-t^{-1})$$

Note that when both k and n are even, then $a(L) = k$, $b(L) = n+1$ and $wr(L) = -n-k+1$, so the factor $(-1)^{wr(L)} t^{\frac{b(L)-a(L)+3wr(L)}{4}}$ reduces to $-t^{\frac{-n}{2}-k+1}$. Now, the Jones polynomial of the link L is

$$V_L = -t^{\frac{-n}{2}-k+1}\left[(-t)^{k-1}\left(\frac{(-t^{k-1})+t}{-t-1}(-t)^{-1}\right)\left(\frac{(-t^{-1})^n - 1}{-t^{-1}-1}\right)\right]$$

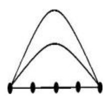
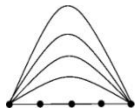
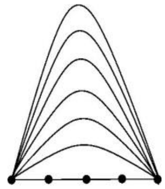
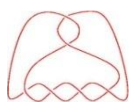
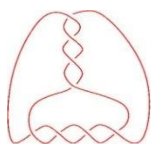
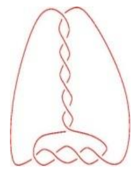
$$= -t^{\frac{-n}{2}-k+1}\left[(-t)^{k-1} + \left(\frac{-t^{k-1}+t}{-t-1} - \frac{1}{t}\right)\left(\frac{\frac{1}{t^n}-1}{-\frac{1}{t}-1}\right)\right]$$

$$= -t^{\frac{-n}{2}-k+1}\left[(-t)^{k-1} + \left(\frac{t^k - t^2 - t - 1}{t(t+1)}\right)\left(\frac{1-t^n}{-t^n(1+t)}\right)\right]$$

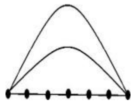
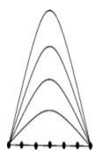
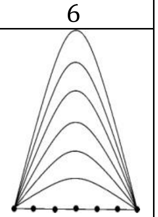
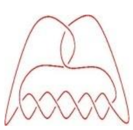
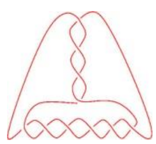
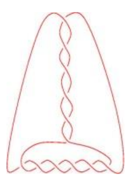
$$= t^{\frac{-n}{2}} + \frac{t^{\frac{-3}{2}n-k+1}}{(1+t)^2}[(t^k - t^2 - t - 1)(1-t^n)]$$

Case II, (When k is odd and n is even**)**

When $k = 5$

n	2	4	6
G			
L			
$a(L)$	5	5	5
$b(L)$	3	5	7
$wr(L)$	-2	0	2

When $k = 7$

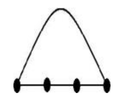
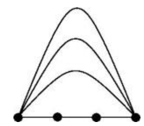
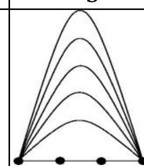
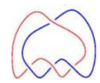
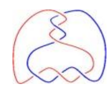
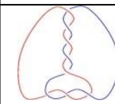
n	2	4	6
G			
L			
$a(L)$	7	7	7
$b(L)$	3	5	7
$wr(L)$	-4	-2	-0

Proposition 3.5: The Jones polynomial of the alternating link L that corresponding to the planar graph $G_{k,n}$, when both k is odd and n is even , is $V_L(t) = t^{\frac{-n}{2}} + \frac{t^{\frac{-3}{2}n-k+1}}{(1+t)^2}[(t^k + t^2 - t - 1)(1 - t^n)]$. *Proof:* This proof is similar to the proof of Proposition 3.4; in this case $a(L) = k, b(L) = n + 1$ and $wr(L) = n - k + 1$ and so the factor reduces to $(-1)^{wr(L)} t^{\frac{b(L)-a(L)+3wr(L)}{4}}$ reduces to $-t^{n-k+1}$

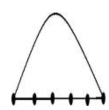
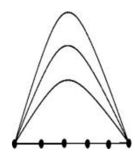
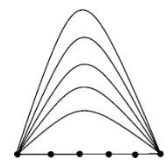
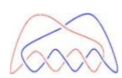
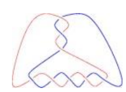
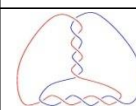
Case III (k *and* n *are odd.***)** In this case we get 1-componet links, and that $a(L) = k, b(L) = n + 1$ and $wr(L) = n + k - 1$.

Proposition 3.6: The Jones polynomial of the alternating link L that corresponding to the planar graph $G_{k,n}$, when both k is odd and n is even , is $V_L(t) = -t^{n+\frac{3k}{2}-1} + \frac{t^{\frac{k}{2}}}{(1+t)^2}[(t^k + t^2 + t + 1)(1 + t^n)]$.

Proof: Similar to the proof of Proposition 3.4**: Case IV.** *(When* k *is even and* n *is odd.)* In this case we receive 2-component links as you can see in the following tables. When $k = 4$

n	1	3	5
G			
L			
$a(L)$	4	4	4
$b(L)$	2	4	6
$wr(L)$	-4,4	-6,6	-8,8

When $k = 6$

n	1	3	5
G			
L			
$a(L)$	6	6	6
$b(L)$	2	4	6
$wr(L)$	-6,6	-8, -8	-10,10

Proposition 3.7: Suppose L is link corresponding to the planar graph $G_{k,n}$ such that k is even and n is odd. If both the component of L are oriented either in clockwise or in counterwise direction then is

$$V_L(t) = -t^{n+\frac{3k}{2}-\frac{3}{2}} + \frac{t^{\frac{k}{2}-\frac{1}{2}}}{(1+t)^2}[(-t^k + t^2 + t + 1)(1 + t^n)].$$

Proof: Similar to the proof of Proposition 3.4 just observe that $a(L) = k, b(L) = n + 1$ and $wr(L) = n + k - 1$, when the components of the link L are oppositely oriented then writhe of L becomes $-n - k + 1$, and we get the result:

Proposition 3.8: Suppose L is link corresponding to the planar graph $G_{k,n}$ such that k is even and n is odd. If both the component of L are oriented either in opposite direction, then

$V_L(t) = -t^{\frac{-n}{2}} - \frac{t^{\frac{-3n}{2}-k+1}}{(1+t)^2}[(-t^k + t^2 + t + 1)(1 + t^n)]$.

The following results reflects the degree of V_L.

Proposition 3.9: Suppose L is the link corresponding to the graph $G_{k,n}$, then

$\deg(V_L) = \frac{1}{4}[n + 3k - 3 + 3wr(L)]$.

Proof: Substitution $x = -t, y - t^{-1}$ in the Equation (3.3) and then simplifying we get

$T(-t,-t^{-1}) = (-t)^{k-1} + \frac{1}{t^n(t+1)}[(-1)^{k-1}t^k + t^2 + t + 1][(-1)^n - t^n]$

Observe that the highest exponent of t in the first term of this equation is $k - 1$ while in the second term is $k - 2$. Since $V_L(t)$ is the product of $(-1)^{wr(L)}t^{\frac{b(L)-a(L)+3wr(L)}{4}}$ and $T(-t,-t^{-1})$, the degree term is actually $\left[(-1)^{wr(L)}t^{\frac{b(L)-a(L)+3wr(L)}{4}}\right][(-t)^{k-1}]$, and hence the degree of $V_L(t)$ is 14[n + 3k − 3 + 3wr(L)]. Hence the degree of V_L is $\frac{1}{4}[n + 3k - 3 + 3wr(L)]$.

3.3. The Flow Polynomial

The flow polynomial was investigated by W. T. Tutte in 1947 in [20] as a function which could count the number of flows in a connected graph.

Definition 3.10. Let G be a graph with an arbitrary but fixed orientation, and let H be an Abelian group of order h and with 0 as its identity element. A $H -$ flow is a mapping φ of the oriented edges $\vec{E}(G))$ into the elements of the group H such that: $\sum_{\vec{e}=u\to v}\varphi(\vec{e}) + \sum_{\vec{e}=u\leftarrow v}\varphi(\vec{e}) = 0$ for every vertex v, and where the first sum is taken over all arcs towards v and the second sum is over all arcs leaving v. A H-flow is nowhere zero if φ never takes the value 0. The relation 4.1 is called the conservation law (that is, the Kirchhoff's law is satisfied at each vertex of G). It is well known [2, 3, 6] that the number of proper H-flows does not depend on the structure of the group, but rather only on its cardinality, and this number is a polynomial function of h that we refer to as the flow polynomial. The following, due to Tutte [22], relates the Tutte polynomial of G with the number of nowhere zero flows of G over a finite abelian group (which, in our case, is $\mathbb{Z}_k$).

Theorem 3.11. [22] Let $G = (V, E)$ be a graph and H a finite Abelian group. If $F_G(h)$ denotes the number of nowhere zero H-flows then

$$F_G(h) = (-1)^{|E|-|V|+k(G)}T(0,1-h).$$

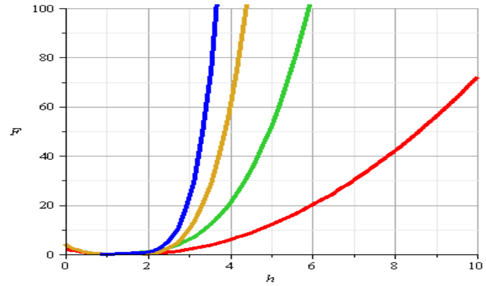
Proposition 3.12. The flow polynomial of the graph $G_{k,n}$ is

$F_{G_{k,n}}(h) = \frac{(-1)^{n+1}}{h}(1-h)[(1-h)^n - 1]$

Proof: We it using Theorem 3.11. Since in the graph $G_{k,n}$, $k(G) = 1, |E| = n + k - 1$ and $|V| = k$, we have

$F_{G_{k,n}}(h) = \frac{(-1)^{n+k-1-k+1}}{h}(0 + 1 - h)[(1-h)^n - 1]$,

which, on simplifying, gives the desired result.



Flow polynomial verses order of the group
The curves for n=2,3,4, and 5 appear from right to left.

3.4. The Chromatic Polynomial

The chromatic polynomial, because of its theoretical and applied importance, has generated a large body of work. Chia [5] provides an extensive bibliography on the chromatic polynomial, and Dong, Koh, and Teo [7] give a comprehensive treatment. For positive integer λ, a $\lambda -$coloring of a graph G is a mapping of $V(G)$ into the set $\{1, 2, 3, \cdots, \lambda\}$ of λ colors. Thus, there are exactly λ^n colorings for a graph on n vertices. If φ is a λ-coloring such that $\varphi(u) \neq \varphi(v)$ for all $uv \in E$, then φ is called a *proper* (or admissible) coloring.

Definition 3.13. The *chromatic polynomial* $P_G(\lambda)$ of a graph G is a one variable graph invariant and is defined recursively by the following deletion contraction relation: $P_G(\lambda) = P(G - e)\ P(G/e)$

In order to find the number of proper $\lambda -$colorings of the graph $G_{k,n}$, we find the chromatic polynomial of this graph as a special case of the Tutte polynomial $T_{G_{k,n}}(x, y)$. The following is the precise relation between these polynomials.

Theorem 3.14. [2] The chromatic polynomial of a graph $G = (V, E)$ is

$P_G(\lambda) = (-1)^{|V|-k(G)}\lambda^{k(G)})T_G(1-\lambda, 0)$, where $k(G)$ denote the number of connected components of G.

Proposition 3.15. The chromatic polynomial of the graph $G_{k,n}$ is

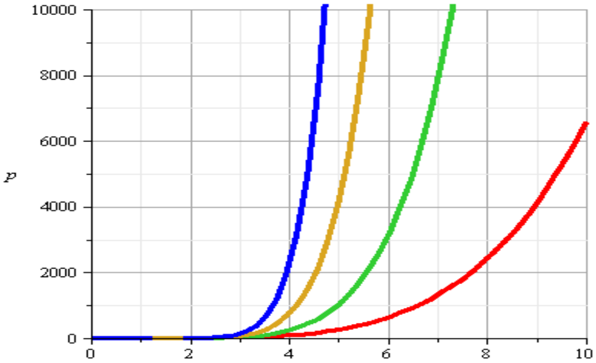
$P_G(\lambda) = (-1)^k\lambda(l - \lambda)[(1-\lambda)^{k-1} - 1]$.

Proof. Note that $|V| = k$ and $k(G) = 1$ for the graph $G_{k,n}$. Now, using Theorem 3.14, we have

$$P_G(\lambda) = (-1)^{k-1}\lambda\left[(1-\lambda)^{k-1} + \frac{(1-\lambda)^{k-1} - (1-\lambda)}{1-\lambda-1}\right]$$
$$= (-1)^{k-1}[\lambda(1-\lambda)^{k-1} - (1-\lambda)^{k-1} + (1-\lambda)]$$
$$= (-1)^{k-1}[(\lambda-1)(1-\lambda)^{k-1} - (\lambda-1)]$$
$$= (-1)^{k-1}(\lambda-1)[(1-\lambda)^{k-1} - 1]$$
$$= (-1)^k(1-\lambda)[(1-\lambda)^{k-1} - 1].$$

3.5. The Reliability Polynomial

Definition 3.16. Let G be a connected graph or network with $|V|$ vertices and $|E|$ edges, and suppose that each edge is independently chosen to be active with probability p. Then the (all terminal) reliability polynomial is



Chromatic polynomial p verses the number of colors λ
(The curves for λ=4,5,6 and 7 appear from right to left.)

$$R_G(p) = \sum_A p^{|A|}(1-p)^{|E-A|}$$

$$= \sum_{i=0}^{|E|-|V|+1} g_i p^{i-|V|-1}(1-p)^{|E|-i-|V|+1}$$

where A is the connected spanning sub-graph of G and gi is the number of spanning connected sub-graphs with $i+|V|-1$ edges. Thus the *reliability polynomial*, $R_G(p)$, is the probability that in this random model there is a path of active edges between each pair of vertices of G.

Theorem 3.17. [8] If G is a connected graph with $|E|$ edges and $|V|$ vertices, then

$$R_G(p) = p^{|V|-1}(1-p)^{|E|-|V|+1} T_G(1, \frac{1}{1-p})$$

Proposition 3.18. The reliability polynomial $R_G(p)$ of $G_{k,n}$ is
$p^{k-1}(1-p)^n + p^{k-2}[(k-2)(1-p)+1][1-(1-p)^n]$.

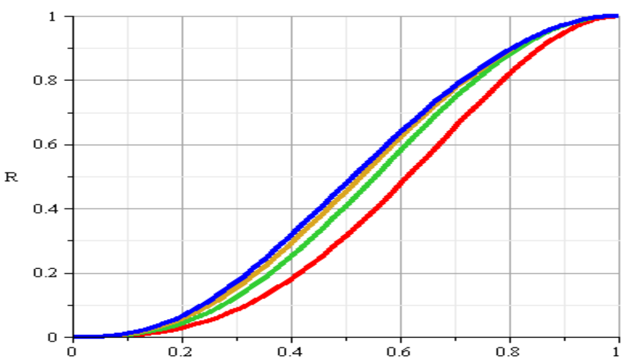
Proof: Since in our case $|V| = k$ and $|E| = n+k-1$, the factor $p^{|V|-1}(1-p)^{|E|-|V|+1}$ becomes $p^{k-1}(1-p)^n$. Also, it is simple to check that $\sum_{i=1}^{k-2} 1^i = k-2$ and $\sum_{j=0}^{n-1}\left(\frac{1}{1-p}\right)^j = \frac{}{p(1-p)^{n-1}}[1-(1-p)]$. Now Using Theorem 3.3 and 3.17, we have

$$R_{G_{k,n}} = p^{k-1}(1-p)^k\left[1+\left(k-2+\frac{1}{1-p}\right)\left(\frac{1}{p(1-p)^{n-1}}\right)(1-(1-p)^n)\right]$$

$$= p^{k-1}(1-p)^n\left[+\left(k-2+\frac{1}{1-p}\right)\left(\frac{1}{p(1-p)^{n-1}}\right)(1-(1-p)^n)\right]$$

$$= p^{k-1}(1-p)^n + p^{k-2}[(k-2)(1-p)+1][1-(1-p)^n],$$

which is the desired result.



Reliability polynomial verses probability
The curves for n=1,2,3, and 4 appear from right to left.

3.6. Subgraphs

The following theorem gives information about the number of different types of sub-graphs of a connected graph G.

Theorem 3.19. [8] If G is a connected graph then:
1. $T_G(1,1)$ is the number of spanning trees of G.
2. $T_G(2,1)$ is the number of spanning forests of G.
3. $T_G(1,2)$ is the number of spanning connected subgraphs of G.
4. $T_G(2,2)$ equals $2|E|$, and is the number of subgraphs of G.

Proposition 3.20. The following statements hold for the connected planar graph $G_{k,n}$.
1. $T_{G_{k,n}}(1,1) = 1 + n(k-1)$
2. $T_{G_{k,n}}(2,1) = 2^{k-1} + n(2^{k-1}-1)$
3. $T_{G_{k,n}}(2,2) = 2^{n+k-1}$.
4. $T_{G_{k,n}}(1,2) = 1 + k(2^n-1)$

Proof. In order to prove it we first note that $\sum_{i=1}^{k-2} 1^i = k-2$, $\sum_{i=1}^{k-2} 2^i = 2(2^{k-2}-1)$, $\sum_{j=0}^{n-1} 1^j = n$, and $\sum_{j=0}^{n-1} 2^j = 2^n - 1$, So
1. $\mathrm{T}(1,1) = (1)^{k-1} + (k-2+1)(n) = 1 + n(k-1)$.
2. $T(2,1) = (2)^{k-1} + [2(2^{k-2}-1)+1](n) = 2^{k-1} + n(2^{k-1}-1)$.
3. $T(2,2) = 2^{n+k-1} since\ |E| = n+k-1$.
4 $.T(1,2) = (1)^{k-1} + [k-2+1](2^n-1) = 1 + k(2^n-1)$.

3.7. Orientations and Score Vectors

The combinatorial interpretations of the Tutte polynomial in Theorem 3.19 are given in terms of the number of certain sub-graphs of the graph G. However, they can also be given in terms of orientations of the graph and its score vectors. An orientation of a graph G is the graph $\vec{G}$ all of whose edges are directed. The score vector of an orientation $\vec{G}$ is the vector $(s_1, s_2, \ldots, s_n)$ such that vertex i has out-degree s_i in the orientation. In the following theorem we gather several similar results about the Tutte polynomial and orientations of a graph.

Theorem 3.21. [8] If G is a connected graph, then
1. $T_G(2,0)$ equals the number of acyclic orientations of G, that is orientations without oriented cycles [4].

2. $T_G(1,0)$ equals the number of acyclic orientations with exactly one predefined source v [24].
3. $T_G(0,2))$ equals the number of totally cyclic orientations of G, that is orientations in which every arc is in a directed cyclic [24].
4. $T_G(2,1)$ equals the number of score vectors of orientations of G [4].

Proposition 3.22. The following statements hold for the connected planar graph $G_{k,n}$
1. $T(2,0) = 2^{k-1}$.
2. $T(0,2) = 2^{n+1} - 2$.
3. $T(1,0) = 1$.
4. $T(2,1) = 2^{k-1} + n(2^{k-1} - 1)$.

Proof. Similar to the proof of Proposition 3.20.

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